Fonctions d'une Variable Complexe

Home Work – 2

Exercice 1

Trouver l'équation du cercle de rayon $r = 2$ et de centre $z_0 = -1 + i$.

Exercice 2

Exprimer chacun des nombres complexes suivants sous la forme polaire.

$$z_1 = 2 - 2\sqrt{3}i,\ z_2 = -3 + 3i \text{ et } z_3 = -\sqrt{6} - \sqrt{2}i.$$

Calculer et représenter

$$\frac{z_1}{z_2},\ z_2 z_3,\ \sqrt{z_3} \text{ et } \sqrt[3]{z_3}.$$

Exercice 3

Résoudre l'équation $z^2 + (-3 + 2i)z + 5 - i = 0$.

Exercice 4

Calculer la limite suivante

$$\lim_{z \to 2e^{-i\pi/2}} \frac{(2z + 1)(z + 2i)}{z^2 + 4}$$

Exercice 5

Trouver la fonction holomorphe $f(z) = u + iv$ telle que

$$u(x, y) = x^3 - 3xy^2$$

Exercise 1

$$r = 2\,;\, z_0 = -1 + i$$

Since the circle is of radius $r = 2$ and center $z_0 = -1 + i$, its equation can be written as

$$|z - z_0|^2 = r^2 = 4$$

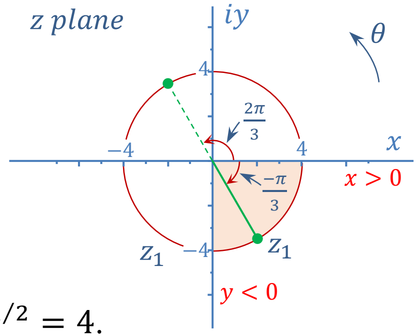
So,

$$|(x - x_0) + (y - y_0)|^2 = r^2 = 4$$

$$(x - x_0)^2 + (y - y_0)^2 = 4$$

$$(x + 1)^2 + (y - 1)^2 = 4$$

Complex Numbers – Polar Form 1



1. $z_1 = 2 - 2\sqrt{3}i$

$z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ with

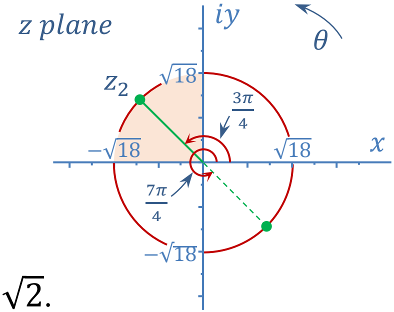
$$r_1 = |z_1| = (x_1^2 + y_1^2)^{1/2},$$

$$r_1 = \left[(2)^2 + \left(-2\sqrt{3}\right)^2\right]^{1/2} = [4 + 12]^{1/2} = 4.$$

$$\tan\theta_1 = \frac{y_1}{x_1} = \frac{-2\sqrt{3}}{2} = -\sqrt{3} > 0 \Longrightarrow \begin{cases}\theta_1 = -\frac{\pi}{3} \\ \theta_1 = \pi - \frac{\pi}{3}\end{cases}.$$

$$\begin{cases} y_1 < 0 \\ x_1 > 0 \end{cases} \Longrightarrow \theta_1 = -\frac{\pi}{3} = 2\pi - \frac{\pi}{3} = \frac{5\pi}{3}.$$

$$z_1 = 4\left[\cos\frac{5\pi}{3} + i\sin\frac{5\pi}{3}\right]. \blacksquare$$



Complex Numbers – Polar Form 2

2. $z_2 = -3 + 3i$

$z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$ with

$$r_2 = |z_2| = (x_2^2 + y_2^2)^{1/2},$$

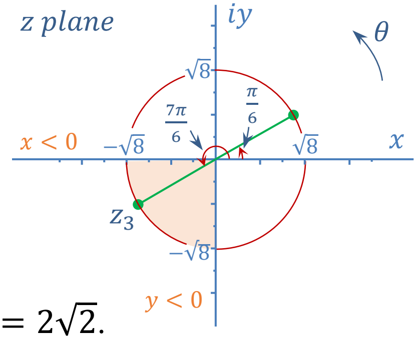
$$r_2 = [(-3)^2 + (3)^2]^{1/2} = [9 + 9]^{1/2} = 3\sqrt{2}.$$

$$\tan\theta_2 = \frac{y_2}{x_2} = \frac{3}{-3} = -1 < 0 \Longrightarrow \begin{cases} \theta_2 \in \left[\frac{\pi}{2}, \pi\right] \\ \theta_2 \in \left[\frac{3\pi}{2}, 2\pi\right] \end{cases} \Longrightarrow \begin{cases} \theta_2 = -\frac{\pi}{4} \\ \theta_2 = \frac{3\pi}{4} \end{cases}.$$

$$\begin{cases} y_2 < 0 \\ x_2 > 0 \end{cases} \Longrightarrow \theta_2 = \frac{3\pi}{4}.$$

$$z_2 = 3\sqrt{2}\left[\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right]. \blacksquare$$

Complex Numbers – Polar Form 3



3. $z_3 = -\sqrt{6} - \sqrt{2}i$

$z_3 = r_3(\cos\theta_3 + i\sin\theta_3)$ with

$$r_3 = |z_3| = (x_3^2 + y_3^2)^{1/2},$$

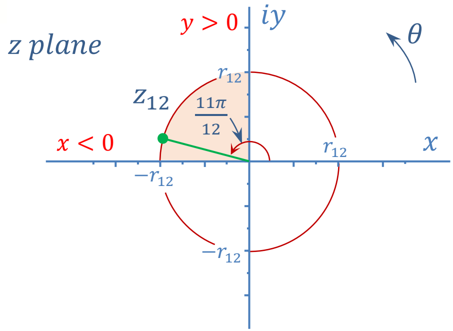
$$r_3 = \left[\left(-\sqrt{6}\right)^2 + \left(-\sqrt{2}\right)^2\right]^{1/2} = [8]^{1/2} = 2\sqrt{2}.$$

$$\tan\theta_3 = \frac{y_3}{x_3} = \frac{-\sqrt{2}}{-\sqrt{6}} = \frac{1}{\sqrt{3}} > 0 \Longrightarrow \begin{cases}\theta_3 = \dfrac{\pi}{6} \\ \theta_3 = \dfrac{\pi}{6} + \pi\end{cases}.$$

$$\begin{cases}y_3 < 0 \\ x_3 < 0\end{cases} \Longrightarrow \theta_1 \in \left[\pi, \frac{3\pi}{2}\right] \Longrightarrow \theta_3 = \frac{\pi}{6} + \pi = \frac{7\pi}{6}.$$

$$z_3 = 2\sqrt{2}\left[\cos\frac{7\pi}{6} + i\sin\frac{7\pi}{6}\right]. \blacksquare$$

Complex Numbers – Polar Form 4



4. $\dfrac{z_1}{z_2} = \dfrac{2-2\sqrt{3}i}{-3+3i}$

$z_{12} = r_{12}(\cos\theta_{12} + i\sin\theta_{12})$ with

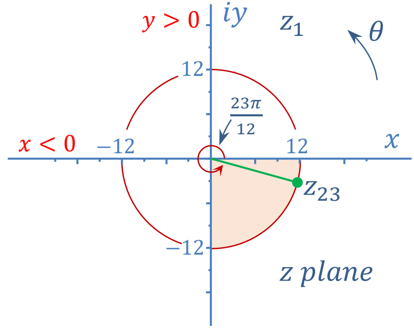
$$r_{12} = \frac{r_1}{r_2} = \frac{4}{3\sqrt{2}} = \frac{2\sqrt{2}}{3}.$$

$$\theta_{12} = \theta_1 - \theta_2 = \frac{5\pi}{3} - \frac{3\pi}{4}.$$

$$\theta_{12} = \frac{5\pi}{3} - \frac{3\pi}{4} = \frac{20\pi}{12} - \frac{9\pi}{12} = \frac{11\pi}{12} = \pi - \frac{\pi}{12}.$$

$$z_{12} = \frac{2\sqrt{2}}{3}\left[\cos\frac{11\pi}{12} + i\sin\frac{11\pi}{12}\right]. \blacksquare$$

Complex Numbers – Polar Form 5



5. $z_2 z_3 = (-3 + 3i)(-\sqrt{6} - \sqrt{2}i)$

$z_2 z_3 = r_{23}(\cos\theta_{23} + i\sin\theta_{23})$ with

$$r_{23} = r_2 r_3 = 3\sqrt{2} \cdot 2\sqrt{2} = 12.$$

$$\theta_{23} = \theta_2 + \theta_3 = \frac{3\pi}{4} + \frac{7\pi}{6} = \frac{9\pi}{12} + \frac{14\pi}{12} = \frac{23\pi}{12}.$$

$$\theta_{23} = \frac{24\pi}{12} - \frac{\pi}{12} = -\frac{\pi}{12} + 2\pi \equiv -\frac{\pi}{12}.$$

$$z_{23} = 12\left[\cos\left(-\frac{\pi}{12}\right) + i\sin\left(-\frac{\pi}{12}\right)\right]. \blacksquare$$

Complex Numbers – Polar Form 6

6. $\sqrt[2]{z_3} = \sqrt[2]{-\sqrt{6} - \sqrt{2}i}$

$$z_3 = 2\sqrt{2}\left[\cos\left(\frac{7\pi}{6} + 2k\pi\right) + i\sin\left(\frac{7\pi}{6} + 2k\pi\right)\right].$$

$$\sqrt{z_3} = \sqrt{2\sqrt{2}}\left[\cos\frac{\frac{7\pi}{6} + 2k\pi}{2} + i\sin\frac{\frac{7\pi}{6} + 2k\pi}{2}\right].$$

$$z_3' \rightarrow k = 0\ ;\ z_3'' \rightarrow k = 1.$$

$$z_3' = \sqrt{2\sqrt{2}}\left[\cos\frac{7\pi}{12} + i\sin\frac{7\pi}{12}\right]. \blacksquare$$

$$z_3'' = \sqrt{2\sqrt{2}}\left[\cos\frac{19\pi}{12} + i\sin\frac{19\pi}{12}\right]. \blacksquare$$

Complex Numbers – Polar Form 7

7. $\sqrt[3]{z_3} = \sqrt[3]{-\sqrt{6} - \sqrt{2}i}$

$$z_3 = 2\sqrt{2}\left[\cos\left(\frac{7\pi}{6} + 2k\pi\right) + i\sin\left(\frac{7\pi}{6} + 2k\pi\right)\right].$$

$$\sqrt[3]{z_3} = \sqrt[3]{2\sqrt{2}}\left[\cos\frac{\frac{7\pi}{6} + 2k\pi}{3} + i\sin\frac{\frac{7\pi}{6} + 2k\pi}{3}\right].$$

$$w_3' \rightarrow k = 0\ ;\ w_3'' \rightarrow k = 1\ ;\ w_3''' \rightarrow k = 2.$$

$$w_3' = \sqrt[3]{2\sqrt{2}}\left[\cos\frac{7\pi}{18} + i\sin\frac{7\pi}{18}\right]. \blacksquare$$

$$w_3'' = \sqrt[3]{2\sqrt{2}}\left[\cos\frac{19\pi}{18} + i\sin\frac{19\pi}{18}\right]. \blacksquare$$

$$w_3''' = \sqrt[3]{2\sqrt{2}}\left[\cos\frac{31\pi}{18} + i\sin\frac{31\pi}{18}\right]. \blacksquare$$

Equation

$$z^2 + (-3 + 2i)z + 5 - i = 0$$

$$\Delta = b^2 - 4ac = (-3 + 2i)^2 - 4 \cdot 1 \cdot (5 - i)$$

$$\Delta = 9 - 4 + 2i(-3)2 - 20 + 4i = -15 - 8i$$

$$\Delta = 1 - 16 - 2 \cdot 4 \cdot i = (1 - 4i)^2$$

So,

$$z_1 = \frac{3-2i+\sqrt{-15-8i}}{2} = \frac{3-2i+(1-4i)}{2} = \frac{4-6i}{2} = 2 - 3i, \blacksquare$$

$$z_2 = \frac{3-2i-\sqrt{-15-8i}}{2} = \frac{3-2i-(1-4i)}{2} = 1 + i. \blacksquare$$

Limit

$$\lim_{z\to 2e^{-i\pi/2}} \frac{(2z+1)(z+2i)}{z^2+4}$$

$$z_0 = 2e^{-i\pi/2} = -2i \Longrightarrow \lim_{z\to -2i} \frac{[2(-2i)+1](-2i+2i)}{(-2i)^2+4} = \frac{0}{0}$$

$$z^2+4 = z^2-(-4) = z^2-(i^2 4) = z^2-(2i)^2 \Longrightarrow$$

$$\lim_{z\to 2e^{-i\pi/2}} \frac{(2z+1)(z+2i)}{z^2+4} = \lim_{z\to -2i} \frac{(2z+1)(z+2i)}{(z+2i)(z-2i)} = \lim_{z\to -2i} \frac{(2z+1)}{(z-2i)}$$

$$\lim_{z\to -2i} \frac{(2z+1)}{(z-2i)} = \frac{2(-2i)+1}{(-2i)-2i} = \frac{-4i+1}{-4i} = 1+i\frac{1}{4}. \blacksquare$$

Function 1 – 1

$$u(x,y) = x^3 - 3xy^2$$

Since $f(z) = u + iv$ is a holomorphic function, its real and imaginary parts $u(x,y)$ and $v(x,y)$ satisfy

The sufficient and necessary Cauchy-Riemann conditions

$$\begin{cases} \dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y} & ① \\ \dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x} & ② \end{cases}$$

Function 1 – 2

$$u(x,y) = x^3 - 3xy^2$$

① $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ with

$$\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}(x^3 - 3xy^2) = 3x^2 - 3y^2 = \frac{\partial v}{\partial y}$$

gives

$$v(x,y) = \int 3x^2 - 3y^2\, dy = 3x^2 y - y^3 + h(x)$$

Function 1 – 3

② $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$ with

$$\frac{\partial u}{\partial y} = \frac{\partial}{\partial y}(x^3 - 3xy^2) = -6xy$$

and

$$-\frac{\partial v}{\partial x} = -\frac{\partial}{\partial x}(3x^2y - y^3 + h(x)) = -6xy - h'(x)$$

$$h'(x) = 0 \Longrightarrow h(x) = c$$

Function 1 – 4

$$f(x,y) = u + iv = (x^3 - 3xy^2) + i(3x^2y - y^3) + ic$$

$$f(x,y) = x(x^2 - 3y^2 + i3xy) + i^3y^3 + ic$$

$$f(x,y) = x(x^2 - y^2 + i2xy) - 2xy^2 + ix^2y + i^3y^3 + ic$$

$$f(x,y) = x(x^2 - y^2 + 2ixy) + 2xi^2y^2 + x^2iy + i^3y^3 + ic$$

$$f(x,y) = x(x^2 - y^2 + 2ixy) + iy(2xiy + x^2 + i^2y^2) + ic$$

$$f(x,y) = x(x^2 - y^2 + 2ixy) + iy(2xiy + x^2 - y^2) + ic$$

$$f(x,y) = (x^2 - y^2 + 2ixy)(x + iy) + ic$$

$$f(z) = z^3 + ic. \blacksquare$$