Harmonic Functions

Exercise

Given $u(x,y)$

- $u(x,y) = 3x^2 - 3y^2 + 5xy + 2$
- $u(x,y) = e^{-x} \sin y$

a harmonic function.

- Find the harmonic function $v(x,y)$ such that u and v are real and imaginary parts of a holomorphic function $f(z) = u + iv$.
- Find this function $f(z)$.

Function 1 – 1

$$u(x,y) = 3x^2 - 3y^2 + 5xy + 2$$

$u(x,y)$ and $v(x,y)$ are real and imaginary parts of a holomorphic function

$$f(z) = u + iv$$

and satisfy the sufficient and necessary Cauchy-Riemann conditions

$$\begin{cases} \dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y} & ① \\ \dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x} & ② \end{cases}$$

Function 1 – 2

$$u(x,y) = 3x^2 - 3y^2 + 5xy + 2$$

① $\dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y}$ with

$$\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}(3x^2 - 3y^2 + 5xy + 2) = 6x + 5y = \frac{\partial v}{\partial y}$$

gives

$$v(x,y) = \int 6x + 5y \, dy = 6xy + \frac{5}{2}y^2 + h(x)$$

Function 1 – 3

② $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$ with

$$\frac{\partial u}{\partial y} = \frac{\partial}{\partial y}(3x^2 - 3y^2 + 5xy + 2) = -6y + 5x$$

and

$$-\frac{\partial v}{\partial x} = -\frac{\partial}{\partial x}\left(6xy + \frac{5}{2}y^2 + h(x)\right) = -6y - h'(x)$$

$$h'(x) = -5x \Longrightarrow h(x) = -\frac{5}{2}x^2 + c$$

Function 1 – 4

$$f(x,y) = u + iv = (3x^2 - 3y^2 + 5xy + 2) + i\left(6xy + \frac{5}{2}y^2 - \frac{5}{2}x^2\right)$$

$$f(x,y) = 3(x^2 - y^2 + i2xy) - (-5xy) + 2 + i\left(\frac{5}{2}y^2 - \frac{5}{2}x^2\right)$$

$$f(x,y) = 3z^2 + 2 + i\left(\frac{5}{2}y^2 - \frac{5}{2}x^2\right) - \left(i^2 2\frac{5}{2}xy\right)$$

$$f(x,y) = 3z^2 + 2 - i\frac{5}{2}(x^2 - y^2 + i2xy)$$

$$f(z) = \left(3 - i\frac{5}{2}\right)z^2 + 2. \blacksquare$$

Function 2 – 1

$$u(x, y) = e^{-x} \sin y$$

$u(x, y)$ and $v(x, y)$ are real and imaginary parts of a holomorphic function

$$f(z) = u + iv$$

and satisfy the sufficient and necessary Cauchy-Riemann conditions

$$\begin{cases} \dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y} & ① \\ \dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x} & ② \end{cases}$$

Function 2 – 2

$$u(x,y) = e^{-x} \sin y$$

① $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ with

$$\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}\left(e^{-x} \sin y\right) = -e^{-x} \sin y = \frac{\partial v}{\partial y}$$

gives

$$v(x,y) = -e^{-x} \int \sin y \, dy = e^{-x} \cos y + h(x)$$

Function 2 – 3

② $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$ with

$$\frac{\partial u}{\partial y} = \frac{\partial}{\partial y}(e^{-x}\sin y) = e^{-x}\cos y$$

and

$$-\frac{\partial v}{\partial x} = -\frac{\partial}{\partial x}[e^{-x}\cos y + h(x)] = e^{-x}\cos y - h'(x)$$

$$h'(x) = 0 \Longrightarrow h(x) = c$$

Function 2 – 4

$$f(x,y) = u + iv = e^{-x}\sin y + ie^{-x}\cos y$$

$$f(x,y) = e^{-x}[\sin y + i\cos y] = ie^{-x}\left[\cos y + \frac{1}{i}\sin y\right]$$

$$f(x,y) = ie^{-x}[\cos y - i\sin y]$$

$$f(x,y) = ie^{-x}e^{-iy} = ie^{-x-iy} = ie^{-(x+iy)}$$

$$f(z) = ie^{-z}. \blacksquare$$