Second Order Linear Differential Equations

Characteristic Equation with Real and Distinct Roots

Exercise 1

Solve the differential equation

$$y'' - 4y' + 3y = 3xe^{x}$$

Characteristic Equation $r^2 - 4r + 3 = 0$. Which gives $r_1 = 3$ and $r_2 = 1$. Hence, $y_H = Ae^{x} + Be^{3x}$. So, $y_p = A(x)\,e^{x}$.

$$C(x) = -\frac{3}{4}(x^2 + x) + c \Longrightarrow y_p = -\frac{3}{4}(x^2 + x)e^{x} + ce^{x}.$$

$$y(x) = y_p + y_H = -\frac{3}{4}(x^2 + x)e^{x} + Ce^{x} + Be^{3x}. \blacksquare$$

Second Order Linear Differential Equations

Characteristic Equation with Coincident Roots

Exercise 2

Solve the differential equation

$$y'' - 4y' + 4y = 0$$

Characteristic Equation $r^2 - 4r + 4 = 0$. Which gives $r_1 = 2$ and $r_2 = 2$. Hence, $y_H = Ae^{2x}$. So, $y_p = A(x)\, e^{2x}$.

$$A(x) = x + c \Longrightarrow y_p = xe^{2x} + ce^{2x}.$$

$$y(x) = y_p + y_H = Be^{2x} + Cxe^{2x}. \blacksquare$$

Second Order Linear Differential Equations

Characteristic Equation with Complex Roots

Exercise 3

Solve the differential equation

$$y'' + 4y' + 13y = 0$$

Characteristic Equation $r^2 + 4r + 13 = 0$, which gives $r = -2 \mp 3i$.
Hence, $y(x) = Ae^{-(2-3i)x} + Be^{-(2+3i)x}$.

$$y(x) = e^{-2x}\{Ae^{3ix} + Be^{-3ix}\} \Longrightarrow y(x) = e^{-2x}\{A_0 \cos 3x + B_0 \sin 3x\}.$$

$$y(x) = Ce^{-2x} \cos(3x - \delta). \blacksquare$$