First Order Differential Equations

Separation of Variables Technique

This technique does not require that the differential equations be linear

Exercise 1

Solve the differential equation $y' = \frac{(x-2)y^4}{4x^3(y^2-3)}$.

Separating variables, we have $\frac{y'(y^2-3)}{y^4} = \frac{(x-2)}{4x^3}$. Which means that $\frac{(y^2-3)}{y^4}dy = \frac{(x-2)}{4x^3}dx$. Hence, $\int \frac{(y^2-3)}{y^4}dy = \int \frac{(x-2)}{4x^3}dx$. So,

$$\frac{1}{y^3} - \frac{1}{y} - \frac{1}{4x^2} + \frac{1}{4x} = C. \blacksquare$$

First Order Differential Equations

Change of Variables

Exercise 2

Solve the differential equation

$$y' = \frac{2y - x}{2x + y}$$

Changing to the variable $s(x) = \frac{y}{x}$, we obtain $s' = \frac{y'}{x} - \frac{y}{x^2}$. Which leads to $s' = \left\{\frac{2s-1}{2+s} - s\right\}\frac{1}{x}$. Or $\frac{2+s}{1+s^2}dz = -\frac{1}{x}dx$. Hence, $\int \frac{2+s}{1+s^2}dz = -\int \frac{1}{x}dx$. So,

$$2\arctan\frac{y}{x} + \frac{1}{2}\ln[x^2 + y^2] = C. \blacksquare$$

First Order Linear Differential Equations

Homogeneous Equations

Exercise 3

Solve the differential equation

$$y' + a(x)\, y = 0$$

The equation is separable and the solution can be written as $\frac{y'}{y} = -a(x)$.

Which means that $\frac{1}{y} dy = -a(x)\, dx$. Hence, $\int \frac{1}{y} dy = -\int a(x)\, dx$. So,

$$\ln y + C = -\int a(x)\, dx$$

$$y = Ae^{-\int a(x)\, dx}. \blacksquare$$

Inhomogeneous First Order Linear Differential Equations

Variation of Parameters Method

Exercise 4

Solve the differential equation

$$3y' + 6xy = x^3$$

We first solve the corresponding homogeneous equation $3y' + 6xy = x^3$, we obtain $y_H = Ae^{-x^2}$. We substitute $y_p = A(x)\,e^{-x^2}$ into the differential equation to write $3\{A'(x)\,e^{-x^2} + A(x)\,(-2x)e^{-x^2}\} + 6x\,A(x)\,e^{-x^2} = x^3$. Hence, $A'(x) = e^{x^2}\,x^3/3$. So, $A(x) = e^{x^2}\,(x^2-1)/6 + C'$

$$y = y_p + y_H = \frac{x^2-1}{6} + C'e^{-x^2} + Ae^{-x^2} = \frac{x^2-1}{6} + Be^{-x^2}.\ \blacksquare$$

Second Order Linear Differential Equations

Homogeneous Equations with Constant Coefficients

Exercise 5

$$ay'' + by' + cy = 0$$

Note that y is a function whose first and second derivatives are proportional to itself. The exponential function $y = Ae^{rx}$ is an obvious solution. Which means that

$$ar^2 + br + c = 0$$

So, the characteristic equation gives two solutions r_1 and r_2:

$$y = Ae^{r_1 x} + Be^{r_2 x}$$