Example 1

Double Integrals

Evaluate the following iterated integrals.

1.

$$\int_2^4 \int_1^3 40 - 2xy \, dx \, dy$$

2.

$$\int_1^3 \int_2^4 40 - 2xy \, dy \, dx$$

Answer: 112

Example 2

Double Integrals

Find the volume of a solid bounded above by the plane $z = 4 - x - y$ and below by the rectangle $[0,1] \times [0,2]$.

1.

$$\int_0^2 \int_0^1 4 - x - y \, dx \, dy$$

2.

$$\int_0^1 \int_0^2 4 - x - y \, dy \, dx$$

Answer: 5 cubic units.

Example 3

Double Integrals

Suppose that the temperature (in degrees Celsius) at a point (x, y) on a flat metal plate is $T(x, y) = 10 + x^2 + 3y^2$, where x and y are in meters.

1. Find the average temperature T_{ave} of the rectangular portion of the plate for which $0 \leq x \leq 1$ and $0 \leq y \leq 2$.

Answer:

$$T_s = \int_0^1 \int_0^2 10 + x^2 + 3y^2 \, dy \, dx = \frac{86}{3}\,°\text{C},$$

$$D = 1 \cdot 2 = 2\,m^2,\ T_{ave} = \frac{T}{2} = \frac{43}{3}\,°\text{C}.$$

Example 4
Double Integrals

Evaluate the following iterated integrals.

$$I = \int_0^{\ln 2} \int_{-1}^{1} \sqrt{e^y + 1} \tan x \, dx \, dy$$

$I = \int_0^{\ln 2} \int_{-1}^{1} \sqrt{e^y + 1} \tan x \, dx \, dy = \int_{-1}^{1} \tan x \, dx \int_0^{\ln 2} \sqrt{e^y + 1} \, dy.$

$\int_{-1}^{1} \tan x \, dx = \int_{-1}^{1} \frac{\sin x}{\cos x} dx = \int_{-1}^{1} \frac{-1}{\cos x} d(\cos x).$

$\int_0^{\ln 2} \sqrt{e^y + 1} \, dy = \int_{\sqrt{2}}^{\sqrt{3}} \frac{2u^2}{u^2-1} du$, with $u = \sqrt{e^y + 1} \Longrightarrow$

$y = \ln(u^2 - 1)$ and $dy = \frac{2u}{u^2-1} du$ hence $\int_0^{\ln 2} \to \int_{\sqrt{2}}^{\sqrt{3}}$.

Answer: 0

Multiple Integrals
Double Integrals

Exercise 1

1. Sketch a diagram of the region $\mathcal{D}$ bounded by the curves $y_1 = \sqrt{x}$ and $y_2 = \sqrt{2(x-1)}$.
2. Evaluate the following iterated integral.

$$\iint_{\mathcal{D}} x - y \, dy \, dx$$

Solution

According to the graph below, we write:

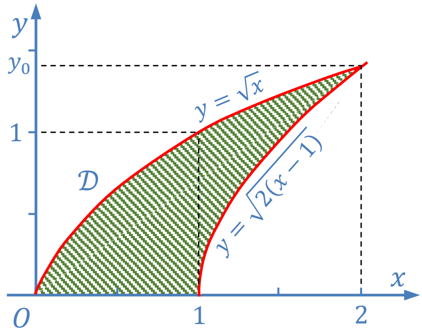
$$\int_0^1 \int_0^{\sqrt{x}} x - y \, dy \, dx + \int_1^2 \int_{\sqrt{2(x-1)}}^{\sqrt{x}} x - y \, dy \, dx$$

The intersection between the two curves occurs when $y_1 = y_2$ at $x_0 = 2, y_0 = \sqrt{2}$.

By changing the order of integration, we get:

$$\int_0^{\sqrt{2}} \int_{y^2}^{(y^2/2)+1} x - y \, dx \, dy$$

with $x_1 = y^2$ and $x_2 = \frac{y^2}{2} + 1$.



Multiple Integrals

Double Integrals

Exercise 2

1. Sketch the region over which the following integration takes place.

$$\int_{1}^{4}\int_{-x+3}^{2x} 2x - 1\, dy\, dx$$

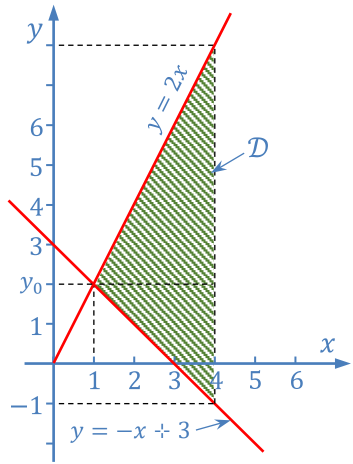
2. Evaluate the iterated integral.
3. Write the equivalent integral by reversing the order of integration.
4. Evaluate the resulting integral.

$$\int_1^4 \int_{-x+3}^{2x} 2x - 1\, dy\, dx = \int_1^4 (2x-1)y\Big|_{-x+3}^{2x} dx,$$

$$= \int_1^4 \{[2x-1]2x - [2x-1][-x+3]\}\, dx,$$

$$= \int_1^4 \{6x^2 - 9x + 3\}\, dx = \left(2x^3 - \frac{9x^2}{2} + 3x\right)\Big|_1^4.$$

$$= \frac{135}{2}.$$



$$\int_{-1}^{2} \int_{-y+3}^{4} 2x - 1\, dx\, dy + \int_2^8 \int_{y/2}^{4} 2x - 1\, dx\, dy,$$

$$= \int_{-1}^{2} (x^2 - x)\Big|_{-y+3}^{4} dy + \int_2^8 (x^2 - x)\Big|_{y/2}^{4} dy,$$

$$= \int_{-1}^{2} \{12 - (y^2 - 6y + 9) - y + 3\}\, dy + \int_2^8 \left\{12 - \frac{y^2}{4} + \frac{y}{2}\right\} dy.$$

$$= \left(6y - \frac{y^3}{3} + \frac{5y^2}{2}\right)\Big|_{-1}^{2} + \left(12y - \frac{y^3}{12} + \frac{y^2}{4}\right)\Big|_2^8 = \frac{135}{2}.$$

Multiple Integrals
Triple Integrals

Exercise 3

Use a triple integral to find the volume of the tetrahedron bounded by the four planes $x = 0$, $z = 0$, $x = 2y$, and $x + 2y + z = 2$.

$$\int_{?}^{?}\int_{?}^{?}\int_{?}^{?} f(x,y,z)\,dz\,dy\,dx$$

$$f(x,y,z) = ?$$

Multiple Integrals

Triple Integrals

Exercise 3

sketch

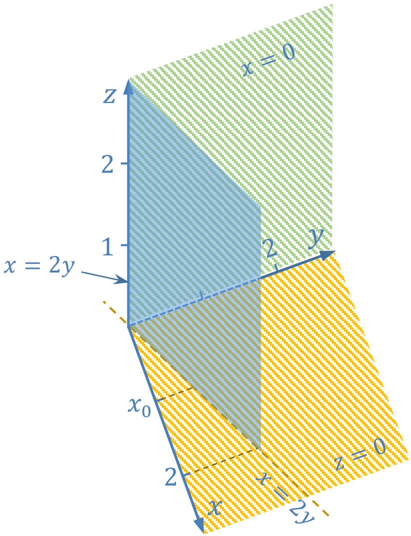
Four planes

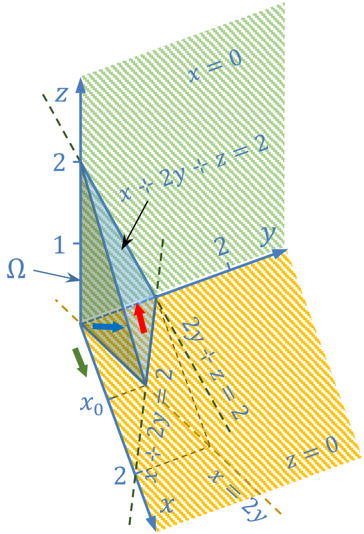
$x = 0,$

$z = 0,$

$x = 2y,$

$x + 2y + z = 2.$





$$\int_{?}^{?}\int_{?}^{?}\int_{?}^{?} f(x,y,z)\,dz\,dy\,dx$$

$$f(x,y,z) = 1$$

Starting with the inner integral, we see that z goes from 0 up to the bottom plane $x + 2y + z = 2 \Longrightarrow$

$$\int_{0}^{2-x+2y} dz$$

Then, y goes from the curve $y = \frac{1}{2}x$ up to the curve $y = 1 - \frac{1}{2}x$.

$$\int_{x/2}^{1-x/2} 2 - x + 2y\,dy$$

x goes from 0 to $x_0 = 1$, the intersection between $x + 2y = 2$ and $x = 2y$.

$$\int_{0}^{1} 3 - 4x + x^2\,dx = \left(3x - 2x^2 + \frac{1}{3}x^3\right)\Big|_0^1 = \frac{4}{3}$$