Complex Numbers

Preliminary

Exercise 1

We define $\mathbb{C}$ as the set of all ordered pairs (a, b) of real numbers a and b. With vector addition and scalar multiplication and complex multiplication defined in $\mathbb{C}$ by:

$(a, b) + (c, d) = (a + c, b + d)$,
$a(b, c) = (ab, ac)$,
$(a, b)(c, d) = (ac - bd, ad + bc)$.

1. Putting $a = (a, 0)$ and $i = (0,1)$, write bi and $a + ib$ and i^2 in this representation.
2. Use the $z = a + ib$ complex number representation, where $\operatorname{Re} z = a$ is the real part and $\operatorname{Im} z = b$ is the imaginary part, and where we define $|z| = (x^2 + y^2)^{1/2}$ to be the absolute value of z and $z^* = a - ib$ the conjugate of z.
   a. Verify that complex multiplication is commutative and associative and distributive.
   b. Show that:
      - If $z \neq 0$ then there exist z' such that $zz' = 1$, and that z' is uniquely determined.
      - z' is the quotient of 1 by z: $z' = z^{-1} = 1/z$.
      - Consequently $1/i = -i$.
   c. Verify the following properties of absolute values and conjugates:
      - $\operatorname{Re} z = \frac{1}{2}(z + z^*)$ and $\operatorname{Im} z = \frac{1}{2i}(z - z^*)$ and $|z|^2 = zz^*$.
      - $(z + w)^* = z^* + w^*$ and $(zw)^* = z^* w^*$.
      - $|zw| = |z||w|$ and $|z/w| = |z|/|w|$ and $|z| = |z^*|$.

Exercise 2

Given the complex numbers $z = x + iy$ and $w = a + ib$, with $\{x, y, a, b\}$ real numbers.

3. Find the real and imaginary parts of each of the following:
   $\frac{1}{z}$; $\frac{z+a}{z-b}$; z^3 ; $\frac{1}{z^2}$; $\frac{3+5i}{7i+1}$; $\left(\frac{-1+i\sqrt{3}}{2}\right)^3$; i^n.
4. Find the absolute value and conjugate of each of the following:
   $-2 + i$; -1 ; $(2 + i)(4 + 3i)$; $\frac{3-i}{\sqrt{2}+3i}$.
5. Prove the following equations:
   d. $|z + w|^2 = |z|^2 + 2 \operatorname{Re} zw^* + |w|^2$.
   e. $|z - w|^2 = |z|^2 - 2 \operatorname{Re} zw^* + |w|^2$.
   f. $|z + w|^2 + |z - w|^2 = 2(|z|^2 + |w|^2)$.
6. Solve the following equations for z:
   $z^2 + i = 1$; $z^4 + i = 1$; $z^4 - i = 1$.
7. Simplify the following expressions:
   $(1 + i)^4$; $(-i)^{-1}$; $(1 - i)^{-1}$; $\frac{1-i}{1+i}$; $\sqrt{1 + \sqrt{i}}$.

Exercise 3

We know that in the complex plane $\mathbb{C}$, every point $z = x + iy$ has polar coordinates (r, θ) such that $x = r\cos\theta$ and $y = r\sin\theta$, where $r = (x^2 + y^2)^{1/2} = |z|$ is called the modulus of z and θ is called the argument of z and is denoted by $\theta = \arg z$. θ is the angle between the positive real axis and the line segment from O to the point z with $0 \leq \theta \leq 2\pi$. Thus, complex numbers, in polar coordinate representation, are written as: $z = x + iy = r(\cos\theta + i\sin\theta)$. Show that:

1. The complex multiplication of $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$ gives $z_1 z_2 = r_1 r_2[\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)]$.
2. Generalize, by induction, the previous result to n complex numbers $z_1 z_2 \cdots z_n$.
3. Deduce de Moivre' formula $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$.
4. Evaluate the cubic root of 1 and i and 8.

Exercise 4

Use De Moivre's formula to reduce the power of $\cos^3 x - \cos^2 x$, then calculate its primitive. Do the same work to find antiderivative of $\sin^4 x$.

Exercise 5

1. To calculate easily the integral

$$\int \frac{dx}{x^4 + 1}$$

we must, at first step, find all the roots of equation $z^4 + 1 = 0$.

2. Calculate

$$\int_0^\pi \cos[\ln(x)]\, dx$$

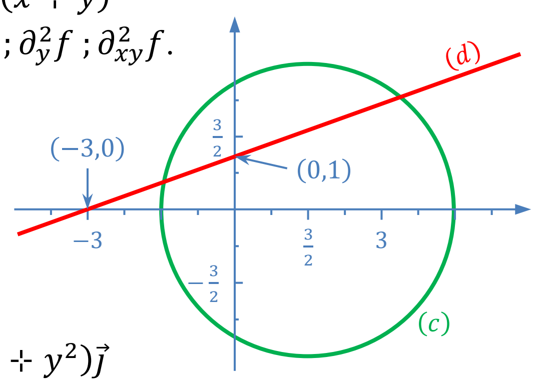
using complex numbers. (Do not use any other method studied in semester 3).

Exercise 6

Let the function f, of two real variables, defined as

$$f(x, y) = x + xy^2 - \cos(x + y)$$

1. Calculate these derivatives of f: $\partial_x f\ ; \partial_y f\ ;\ \partial_x^2 f\ ; \partial_y^2 f\ ; \partial_{xy}^2 f$.
2. Deduce its total differential df.
3. Is f harmonic?



Exercise 7

Let $\vec{V}$ be a vector field in the plane defined as

$$\vec{V}(x, y) = y\vec{\imath} - (x^2 - 3x + y^2)\vec{\jmath}$$

Choose a convenient scale and sketch $\vec{V}$ at points $A(1,0)$ and $B(2,3)$ and $C(0,4)$ and $D(-3,-1)$.

Describe $\vec{V}$ on the straight line (d) and the circle (c).

Integrate the field $\vec{V}$ over (d) from $(-3,0)$ to $(0,1)$ and over (c).